

ENERGY SOLUTIONS TO ONE-DIMENSIONAL SINGULAR PARABOLIC PROBLEMS WITH BV DATA ARE VISCOSITY SOLUTIONS

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ABSTRACT. We study one-dimensional very singular parabolic equations with periodic boundary conditions and initial data in BV , which is the energy space. We show existence of solutions in this energy space and then we prove that they are viscosity solutions in the sense of Giga-Giga.

Key words: singular energies, viscosity solutions

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1. INTRODUCTION

We study a Cauchy problem for one-dimensional singular parabolic equations of the form

$$(1.1) \quad \partial_t u = (\mathcal{L}(u_x))_x, \quad (x, t) \in Q_T := \mathbb{T} \times (0, T),$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{T},$$

where $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$ is monotone increasing and bounded. Such equations arise in models of crystal growth or image analysis. This is why it is reasonable to assume that $u_0 \in BV$. For the sake of simplicity, we consider periodic boundary conditions. Here, \mathbb{T} is the one-dimensional flat torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Since \mathcal{L} is merely monotone increasing, it may have jumps. Typical examples are

$$(1.3) \quad \partial_t u = (\operatorname{sgn}(u_x))_x,$$

and

$$(1.4) \quad \partial_t u = (\operatorname{sgn}(u_x - 1) + \operatorname{sgn}(u_x + 1))_x.$$

Equation (1.3) is a prototype for the class, we have in mind, (1.1). We also point out that equation (1.3) can be formally written as $\partial_t u = 2\delta(u_x)u_{xx}$, where δ is the Dirac delta function.

Since problem (1.1) is one-dimensional, then for a given increasing \mathcal{L} , we can always find convex W , which we will call the energy density such that

$$\mathcal{L}(p) = \frac{dW}{dp}(p), \quad \text{for a.e. } p \in \mathbb{R}.$$

Thus, we can write (1.1) as an L^2 -gradient flow

$$(1.5) \quad \partial_t u = -\nabla E[u]$$

with the corresponding energy, defined by

$$(1.6) \quad E[u] = \begin{cases} \int_{\mathbb{T}} W(Du) & \text{if } u \in BV(\mathbb{T}), \\ +\infty & \text{if } u \in L^2(\mathbb{T}) \setminus BV(\mathbb{T}). \end{cases}$$

In order to avoid unnecessary technicalities at this stage, we will restrict our attention to piecewise linear W 's which are coercive, i.e. $W(p) \rightarrow \infty$ when $|p| \rightarrow \infty$. We shall see (cf. (2.3)) that indeed $E[u]$ is well-defined. We will look for solutions for which $t \mapsto E[u(t)]$ is bounded over $[0, T]$, we will call them *energy solutions*. There is substantial literature on functionals on measures, including [7], [28] and others. The general case of W requires additional considerations, see [2], [7], [14], [28].

In case of (1.3), we have $W(p) = |p|$, this is why it is called the *total variation flow*. Equation (1.3) has been studied quite extensively, [15], [19], [24], [25], [30], [32], [35]. These authors did not allow jumps in the initial datum u_0 . For the sake of completeness, we should mention the growing literature on the higher dimensional versions of (1.3). Let us mention only [3], [9], [4], [5], [6], [21].

On the other hand, (1.4) was studied in [29] with initial condition $u_0 \in BV$. There, they showed that if $u_0 \in BV$, then the unique solution u , which we constructed there, belongs to $L^\infty(0, T; BV)$, see [29, Theorem 2.1]. Moreover, this kind of regularity is optimal, because there are discontinuous solutions.

In case of (1.4), we have $W(p) = |p + 1| + |p - 1|$ then, we define $E[u]$ for $u \in BV$ in a natural way, introduced in [29], which can be easily extended to convex piecewise linear W . However, for the sake of simplifying the exposition, we will use a piecewise linear structure of W , generalizing the method used in [29]. A difficulty here is the composition of nonlinear function \mathcal{L} with a measure Du , when $u \in BV$.

We also showed in [29] that energy solutions are viscosity solutions in the sense of [23]. However, the proof of this fact presented in [29] was direct and involved. Here, we would like to generalize this fact. However, in order to develop intuition we will restrict our attention to the case of convex, piecewise linear and coercive W . This will be a necessary preparation before tackling the general case of W with linear growth. The direct method of ([29]) is not likely to work.

We will show that problem (1.1–1.2) is well posed for $u_0 \in BV$, which is the energy space. We will exploit for this purpose the gradient flow structure of (1.1). We will construct solutions by means of regularizations as in [32] and [29]. This will be done in Theorem 2.5 below. However, our main result is Theorem 4.1 stating that energy solutions to (1.1) are also viscosity solutions in the sense of [23]. With this result, we obtain new tools to study properties of solutions to (1.1). Incidentally, this theorem shows that for regular data, i.e. when not only u_0 is in BV but also $u_{0,x} \in BV$, then the notion of energy solutions, the almost classical solutions introduced in [31] and the viscosity solutions to (1.1) coincide.

The rationale behind our present work is the usefulness of viscosity solutions. Originally, the viscosity solution theory was introduced in early 1980s by Crandall and Lions, [12] as a weak solution of first-order Hamilton-Jacobi equation and generalized to some regular second-order equations around 1990; see [11]. A big success of this theory was the study of the mean curvature flow equation independently by Evans-Spruck [13] and Chen-Giga-Goto [10]. The viscosity solution for the very singular diffusion equation is first introduced by Giga-Giga [16, 17, 18]. The authors establish a theorem on a unique existence of the viscosity solution of a class of very singular diffusion equations including (1.1) with continuous initial data u_0 .

The present paper provides more of examples existence theorems, this time with discontinuous initial conditions.

The plan of the paper is as follows: in Section 2, we show a lower semicontinuity of functional E , then existence of solutions in the energy space by the regularization method. Section 3 is devoted to introducing the necessary notions pertinent to the viscosity solutions. In Section 4, we construct monotone sequences of continuous functions approximating the initial conditions. This guarantees the existence of continuous solutions to (1.1)–(1.2). Then, passing to the limit with the regularizing parameter yields the desired result.

2. ENERGY SOLUTIONS

We could use the abstract semigroup theory, see [8], exploiting the gradient flow structure, (1.5), to prove existence and uniqueness of solutions to (1.2) with data in L^2 or BV . For this purpose, it suffices to show that E defined by (1.6) is convex, proper and lower semicontinuous with respect to the L^2 topology. However, in this paper, not only the fact that u_0 belongs to BV is important but also how u_0 is approximated by smooth functions. This is why we will use the method of regularization to construct energy solutions. By ‘energy solutions’ we will understand weak solutions with data from the vector space where E is finite. We restrict our attention to W which are piecewise linear, convex and coercive.

2.1. Functional E . Understanding E is crucial for the present paper. We consider only W given by

$$(2.1) \quad W(p) = \sum_{j=1}^{N^+} \alpha_j^+ (p + \beta_j^+)^+ + \sum_{j=1}^{N^-} \alpha_j^- (p + \beta_j^-)^-,$$

where N^\pm are positive integers, all α_i^\pm are positive and β_i^\pm are real. We adopt the usual definition, $x^+ = \max\{0, x\}$ and $x^- = (-x)^+$ for a real number x .

First, we will explain that these are all functions of interest to us.

Proposition 2.1. *Let us suppose that $W : \mathbb{R} \rightarrow \mathbb{R}$ is piecewise linear, convex and there is $M > 0$ such that $\frac{dW}{dp}(p)$ is constant for $|p| > M$. Then, W has the form (2.1).*

Proof. We will first find β_i^\pm ’s. We notice that coercivity and convexity of W imply that

$$a_0 := \min\{x : W(x) = \min\{W(p) : p \in \mathbb{R}\}\}$$

is well defined. Since by assumption W is linear for large arguments, then by definition, there are real a_i , $i = -M, \dots, N$, (we also set $a_{N+1} = +\infty$) such that

$$\begin{aligned} W(p) &= s_i(x - a_i) + W(a_i), & \text{for } x \in [a_i, a_{i+1}), \quad i = -M, \dots, N+1, \\ W(p) &= s_{-M-1}(x - a_{-M}) + W(a_{-M}) & \text{for } x \in (-\infty, a_{-M}). \end{aligned}$$

If $s_0 = 0$, then we set

$$\begin{aligned} \beta_i^+ &= a_i, \quad i = 1, \dots, N, \quad \text{and} \quad N^+ := N, \\ \beta_i^- &= a_{1-i}, \quad i = 1, \dots, M, \quad \text{and} \quad N^- := M. \end{aligned}$$

Otherwise, we define

$$\begin{aligned} \beta_i^+ &= a_{i-1}, \quad i = 1, \dots, N+1, \quad \text{and} \quad N^+ := N+1, \\ \beta_i^- &= a_{1-i}, \quad i = 1, \dots, M, \quad \text{and} \quad N^- := M. \end{aligned}$$

Monotonicity of the derivative of W implies monotonicity of sequence s_i , $i = -M, \dots, N$. We set $\alpha_1^+ = s_1$. Since $s_{i+1} > s_i$, then there is positive α_{i+1} such that

$$s_{i+1} = \alpha_{i+1}^+ + s_i = \sum_{k=1}^i \alpha_k^+, \quad i = 1, \dots, N^+ - 1.$$

Similarly, we define α_i^- , $i = 1, \dots, N^-$. \square

Formula (2.1) and positivity of N^+ and N^- imply the existence of positive c_0, c_1, c_2 such that

$$(2.2) \quad c_1|p| - c_2 \leq W(p) \leq c_0(|p| + 1).$$

The advantage of (2.1) is clearly seen when it comes to defining $\int_{\mathbb{T}} W(Du)$, because it refers to well-defined operations on measures. Indeed, if μ is a (finite) signed measure, then $\mu^\pm = \frac{1}{2}(|\mu| \pm \mu)$, where $|\mu|$ is the variation of μ . Thus, for $u \in BV(\mathbb{T})$, the following formula is correct, despite the lack of homogeneity of W ,

$$(2.3) \quad \int_{\mathbb{T}} W(Du) = \int_{\mathbb{T}} \left(\sum_{i=1}^{N^+} \alpha_i^+ (Du + \beta_i^+)^+ + \sum_{j=1}^{N^-} \alpha_j^- (Du + \beta_j^-)^- \right).$$

We notice that obviously E is convex. It is also lower semicontinuous.

Proposition 2.2. *E is lower semicontinuous with respect to the convergence in the L^2 topology.*

Proof. Let us suppose that $u_n \rightarrow u$ in L^2 . Since for any $v \in BV(\mathbb{T})$ and $a \in \mathbb{R}$, we have

$$\int_{\mathbb{T}} (Dv + a) = a|\mathbb{T}|,$$

then in order to prove our claim, it is sufficient to check that

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{T}} |Du_n + a| \geq \int_{\mathbb{T}} |Du + a|.$$

We may assume $\int_{\mathbb{T}} |Du_n| \leq M < \infty$, $n \in \mathbb{N}$, for otherwise there is nothing to prove. This bound and the lower semicontinuity of the BV norm imply

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{T}} |Du_n| \geq \int_{\mathbb{T}} |Du|,$$

as a result, $u \in BV$.

We recall that if $u \in BV(\mathbb{T})$, then

$$Du = \frac{du}{dx} \llcorner \mathcal{L}^1 + D^c u + \sum_{i \in J} a_i \delta_{x_i},$$

where $\frac{du}{dx} \llcorner \mathcal{L}^1$ is absolutely continuous with respect to \mathcal{L}^1 part of Du , $D^c u$ is the continuous part singular with respect to \mathcal{L}^1 and $\sum_{i \in J} a_i \delta_{x_i}$ is the jump part. Since the number of jump discontinuities points of u and u_n , $n \in \mathbb{N}$, is at most countable, then there exists $x_0 \in \mathbb{T}$, a common continuity point of all u_n , $n \in \mathbb{N}$ and u . That is,

$$(2.4) \quad |Du_n|(\{x_0\}) = 0 = |Du|(\{x_0\}) \quad \text{for all } n \in \mathbb{N}.$$

Once we identify \mathbb{T} with $[0, 1)$, then for the sake of simplicity of notation we may assume that $x_0 = 0$. We notice that

$$Du_n + a = D(u_n + a) - a\delta_0.$$

Furthermore, due to (2.4), we have

$$|D(u_n + a) - a\delta_0| = |D(u_n + ax)| \llcorner \mathbb{T} \setminus \{0\}.$$

We notice that $(u_n + ax)\chi_{\mathbb{T} \setminus \{0\}}$ is a bounded sequence in $BV(\mathbb{T})$, hence the lower semicontinuity of the norm implies that

$$\liminf_{n \rightarrow +\infty} \int_{\mathbb{T}} |D(u_n + ax)| \geq \int_{\mathbb{T}} |D(u + ax)|.$$

□

Remark 2.3. Density W is not homogeneous, but since W is piecewise linear, one could easily see that E coincides with its relaxation, cf. [2], [14], [28].

2.2. Existence of energy solutions by the regularization method. In this section our goal is to examine the existence of solutions to (1.1), (1.2), when the nonlinearity \mathcal{L} is the derivative of a convex function W , given by (2.1). We studied problem (1.1–1.2) in [32], for rather general nonlinearities \mathcal{L} and Dirichlet boundary conditions, while assuming that $u_0 \in L^1$ and $u_{0,x} \in BV$. In [29] we considered general data $u_0 \in BV$, while focusing on special \mathcal{L} ,

$$\mathcal{L}(p) = \operatorname{sgn}(p + 1) + \operatorname{sgn}(p - 1).$$

Here, we will use the method of [32] and [29] to prove the existence of energy solutions.

Of course, one could study W with a different growth rate at infinity, but we will not do this here. We would be happy to work with general W , but for the time being we will consider only W which is convex and piecewise linear, given by (2.1). In principle, one of N^+ and N^- could be zero, but we will not address this issue here.

We introduce the definition of solutions to (1.1) with periodic boundary conditions. In principle, it is well-known, see [3], [32].

Definition 2.4. We shall say that function $u \in L^2(0, T; L^2(\mathbb{T}))$ is a *finite energy* solution to (1.1) if $u \in L^\infty(0, T; BV(\mathbb{T}))$ and $u_t \in L^2(0, T; L^2(\mathbb{T}))$ and there is $\Omega \in L^2(0, T; W^{1,2}(\mathbb{T}))$ satisfying the identity

$$(2.5) \quad \langle u_t, \varphi \rangle = - \int_{\mathbb{T}} \Omega \varphi_x dx$$

for all test functions $\varphi \in C^\infty(\mathbb{T})$ and for almost every $t > 0$.

We notice that the time regularity, postulated in Definition 2.4 implies that solutions to (1.1) are in $C([0, T]; L^2(\mathbb{T}))$. Hence, we can impose initial conditions (1.2). At this point we stress that Ω is a selection of the composition of multivalued operators $\mathcal{L} \circ u_x$.

We will show the existence result. We note that we consider less regular initial conditions than in [32]. We prefer to use tools based on approximation of (1.1) by smooth problems.

Theorem 2.5. *Let us suppose that $u_0 \in BV(\mathbb{T})$, then there exists an energy solution to (1.1-1.2). Moreover, since for almost all $t > 0$, u satisfies*

$$(2.6) \quad \int_{\mathbb{T}} [W(u_x + h_x) - W(u_x)] dx \geq \int_{\mathbb{T}} \Omega h_x dx,$$

for all $h \in C^\infty(\mathbb{T})$, hence u is a unique solution.

Proof. Step 1. After regularizing \mathcal{L} and u_0 , we obtain a uniformly parabolic problem,

$$(2.7) \quad \begin{aligned} \frac{\partial u^\epsilon}{\partial t} &= (\mathcal{L}^\epsilon(u_x^\epsilon))_x, & (x, t) \in Q_T, \\ u^\epsilon(x, 0) &= u_0^\epsilon(x), & x \in \mathbb{T}, \end{aligned}$$

where ϵ is a regularizing parameter. We define W^ϵ by the formula below,

$$(2.8) \quad W^\epsilon(p) = \int_0^p \mathcal{L}^\epsilon(s) ds + \frac{\epsilon}{2} p^2.$$

We notice that

$$(2.9) \quad W(p) \leq W^\epsilon(p) \leq W(p) + k\epsilon + \frac{\epsilon}{2} p^2,$$

where k is chosen in an appropriate way.

By the classical theory, see [26], we obtain existence and uniqueness of smooth solutions to (2.7).

If we multiply (2.7) by u_t^ϵ and integrate over Q_T , then we reach,

$$\int_0^T \int_{\mathbb{T}} (u_t^\epsilon)^2 dx dt = \int_0^T \int_{\mathbb{T}} (\mathcal{L}^\epsilon(u_x^\epsilon))_x u_t^\epsilon dx dt.$$

Integration by parts yields,

$$\int_0^T \int_{\mathbb{T}} (u_t^\epsilon)^2 dx dt = - \int_0^T \int_{\mathbb{T}} \mathcal{L}^\epsilon(u_x^\epsilon) u_{xt}^\epsilon dx dt = - \int_0^T \int_{\mathbb{T}} \frac{d}{dt} W^\epsilon(u_x^\epsilon) dx dt.$$

We identify \mathbb{T} with the interval $[0, 1)$. We have to justify, why the boundary terms drop out in the formula above. For this purpose, we recall a well-known result, see e.g. [27].

Lemma 2.6. *If $w \in H^1(a, b)$ and $\varphi \in BV(a, b)$, then*

$$\int_a^b w_x \varphi dx = - \int_a^b w D\varphi + \gamma(w\varphi)|_a^b. \quad \square$$

Due to the periodicity of the ingredients, the integral over (a, b) equals the integral over $(a + \delta, b + \delta)$ and we can select such δ that $x = a + \delta$, $x = b + \delta$ are points of continuity of $u - v$, as considered over \mathbb{R} .

Hence, we reach the following conclusion,

$$(2.10) \quad \int_0^T \int_{\mathbb{T}} (u_t^\epsilon)^2 dx dt + \int_{\mathbb{T}} W^\epsilon(u_x^\epsilon(x, T)) dx = \int_{\mathbb{T}} W^\epsilon(u_{0,x}^\epsilon) dx.$$

Now, we will pass to the limit. First of all, we notice that (2.9) implies

$$\int_{\mathbb{T}} W^\epsilon(u_{0,x}^\epsilon) \leq k\epsilon|\mathbb{T}| + \sup_{\epsilon \in [0, 1]} \int_{\mathbb{T}} W(u_{0,x}^\epsilon) + \int_{\mathbb{T}} \frac{\epsilon}{2} |u_{0,x}^\epsilon|^2 dx =: M < +\infty,$$

where we used the following inequality,

$$\frac{\epsilon}{2} \int_{\mathbb{T}} |u_{0,x}^\epsilon|^2 dx \leq C \|u_0\|_{BV}^2.$$

Since we have found a bound on the right-hand-side (RHS) of (2.10) independent of ϵ , we conclude that

$$\int_0^T \int_{\mathbb{T}} (u_t^\epsilon)^2 \leq M \quad \text{and} \quad \sup_{t \in [0, T]} \int_{\mathbb{T}} W^\epsilon(u_x^\epsilon(x, t)) \leq M.$$

Thus, we can select a subsequence $\{u^\epsilon\}$ such that

$$u^\epsilon \rightharpoonup u \quad \text{in} \quad L^2(0, T; L^2(\mathbb{T}))$$

and

$$u_t^\epsilon \rightharpoonup u_t \quad \text{in} \quad L^2(0, T; L^2(\mathbb{T})).$$

Moreover, if we set $B = W^{1,1}(\mathbb{T})$, $X = Y = L^2(\mathbb{T})$, then the above estimates show that family $\{u^\epsilon\}$ is bounded in $L^\infty(0, T, B) \cap L^1(0, T; X)$ and $\{\frac{\partial u^\epsilon}{\partial t}\}$ is bounded in $L^1(0, T; Y)$. Since BV is compactly embedded in L^2 , then by Aubin Lemma (see [34, Corollary 6]), we can select a subsequence $\{u^\epsilon\}$ such that

$$u^\epsilon \rightarrow u \quad \text{in} \quad L^p(0, T, L^q(\mathbb{T})),$$

where p, q are arbitrary from the interval $(1, \infty)$. As a result for almost all $t \in (0, T)$

$$\|u^\epsilon(\cdot, t) - u(\cdot, t)\|_{L^q} \rightarrow 0.$$

Thus, we use the lower semicontinuity of E , see Proposition 2.2, we deduce that for almost all $t > 0$

$$\liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}} W^\epsilon(u_x^\epsilon)(x, t) dx \geq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{T}} W(u_x^\epsilon) dx \geq \int_{\mathbb{T}} W(Du)(\cdot, t).$$

Combining these inequalities, we arrive at

$$(2.11) \quad \int_0^t \int_{\mathbb{T}} u_t^2(x, s) dx ds + \int_{\mathbb{T}} W(Du)(\cdot, t) dx \leq M \quad \text{for almost all } t \in (0, T).$$

Moreover, due to (2.2), we have a bound on the BV norm of $u(\cdot, t)$,

$$\int_{\mathbb{T}} |Du| \leq C(1 + \int_{\mathbb{T}} W(Du)).$$

We also have to indicate a candidate for Ω , as required by the definition of a solution. We set

$$\Omega^\epsilon(x, t) := \mathcal{L}^\epsilon(u^\epsilon(x, t)).$$

Since $u_t^\epsilon = \Omega_x^\epsilon$, then due to (2.10), we deduce that

$$(2.12) \quad \|\Omega^\epsilon\|_{L^2(0, T; H^1(\mathbb{T}))} \leq M_1 < +\infty.$$

Hence, we can select a subsequence,

$$\Omega^\epsilon \rightharpoonup \Omega \quad \text{in} \quad L^2(0, T; H^1(\mathbb{T})).$$

Moreover,

$$\int_0^T \int_{\mathbb{T}} u_t \varphi dx dt = \int_0^T \int_{\mathbb{T}} \Omega_x \varphi dx dt \quad \text{for all } \varphi \in C_0^\infty((0, T) \times I).$$

At this point, we may apply [32, Lemma 2.1] to conclude that (2.5) holds. Moreover, [32, Lemma 2.2] implies (2.6).

Step 2. We shall establish uniqueness of solutions. We notice that if u is a solution to (1.1), according to Definition 2.4, then $t \mapsto u(t) \in L^2(\mathbb{T})$ is continuous. In particular it makes sense to evaluate u at $t = 0$.

Let us suppose that u and v are solutions to (1.1) satisfying $u(0) = u_0 = v(0)$. Since we identify \mathbb{T} with $[0, 1)$, we notice that for any $\delta \in (0, 1)$. We have

$$(2.13) \quad \int_0^1 (\Omega_x(u) - \Omega_x(v)) (u - v) dx = \int_\delta^{1+\delta} (\Omega_x(u) - \Omega_x(v)) (u - v) dx.$$

Moreover, $\Omega(u) - \Omega(v)$ belongs to $H^1(\mathbb{T})$, while $u - v$ is a function from $BV(\mathbb{T})$. We want to perform integration by parts in the RHS of (2.13). For this purpose, we use Lemma 2.6 with $\delta > 0$ so chosen that the boundary terms drop out due to continuity of the integrand at δ . Thus, we conclude that

$$\frac{1}{2} \|u - v\|_{L^2(I)}^2(T) = - \int_0^T \int_{I+\delta} (\Omega(u; x, t) - \Omega(v; x, t)) (u_x - v_x) dx dt.$$

On the other hand, monotonicity of \mathcal{L} yields

$$(2.14) \quad \frac{1}{2} \|u - v\|_{L^2(\mathbb{T})}^2(\tau) \leq 0.$$

We conclude that $u = v$, as desired. \square

2.3. Regularity of energy solutions. We recall the following fact, which will be very useful for us.

Proposition 2.7 ([32]). *Let us suppose that $u_0 \in BV$ as well as $u_{0,x} \in BV$. Then there is $\alpha > 0$ such that $u \in C^{2\alpha, \alpha}$.*

We refer the reader to [32, formula (2.19)].

2.4. Stability of energy solutions. We show a very useful result for further studies.

Proposition 2.8. *Let us suppose that (u^n, Ω^n) is a sequence of energy solutions to (1.1–1.2), which is bounded in L^∞ , u_n converges to u almost everywhere and $\Omega^n \rightharpoonup \Omega$ in $L^2(0, T; H^1(I))$. Then, u and Ω form an energy solution to (1.1–1.2) with initial data u_0 .*

Proof. Essentially, this follows from our definition of energy solutions and [32, Lemma 2.1]. \square

3. VISCOSITY SOLUTIONS

In this section, we review the viscosity solution theory to the singular diffusion equation of the generalized form

$$(3.1) \quad \partial_t u + F((W'(u_x))_x) = 0 \quad \text{in } U_T := (0, T) \times U.$$

Here, U is an open set in \mathbb{T} and F is a function such that

- (F1) (Continuity) $F \in C(\mathbb{R})$,
- (F2) (Ellipticity) F is non-increasing.

We refer the readers to the papers [23, 20] for the details.

Let \mathbb{R} denote extended real numbers $\mathbb{R} \cup \{\pm\infty\}$, and for an \mathbb{R} -valued function u on U_T define its upper (resp. lower) semicontinuous envelope u^* (resp. u_*) by

$$u^*(t, x) := \overline{\lim}_{(s, y) \rightarrow (t, x)} u(s, y) \quad (\text{resp. } u_*(t, x) := \underline{\lim}_{(s, y) \rightarrow (t, x)} u(s, y)).$$

3.1. The notion of viscosity solutions to (1.1). We recall a notion of faceted functions. Let P denote the set of the jump points of the derivative of W . A function $f \in C^1(U)$ is *faceted* at a point $\hat{x} \in U$ with *slope* $p \in \mathbb{R}$ (or *p-faceted* at \hat{x}) if there exists a closed nontrivial finite interval $I = [c_l, c_r] \subset U$ containing \hat{x} (i.e. $c_l, c_r \in U$ satisfy $c_l < c_r$ and $c_l \leq \hat{x} \leq c_r$) such that

$$\begin{aligned} f'(x) &= p \quad \text{for all } x \in I, \\ f'(x) &\neq p \quad \text{for all } x \in J \setminus I \end{aligned}$$

for a neighborhood $J = (b_l, b_r) \subset U$ of I . The closed interval I is called a *faceted region* of f containing \hat{x} . Let $C_P^2(U)$ denote the set of all $f \in C^2(U)$ such that f is p -faceted at \hat{x} whenever $p = f'(\hat{x}) \in P$. We also define the *left transition number* $\chi_l = \chi_l(f, \hat{x})$ and the *right transition number* $\chi_r = \chi_r(f, \hat{x})$ for a p -faceted function f at \hat{x} by

$$\chi_l = \begin{cases} +1 & \text{if } f' < p \text{ on } (b_l, c_l), \\ -1 & \text{if } f' > p \text{ on } (b_l, c_l), \end{cases} \quad \chi_r = \begin{cases} +1 & \text{if } f' > p \text{ on } (c_r, b_r), \\ -1 & \text{if } f' < p \text{ on } (c_r, b_r). \end{cases}$$

Now, for $\Delta > 0$, $I \subset U$, $\xi_l, \xi_r \in [-1, 1]$ and $Z \in C^{1,1}(I)$, we consider the obstacle problem to minimize

$$J[\xi] := \begin{cases} \int_I |\xi'(x)|^2 dx & \text{if } \xi \in H^1(I), \\ \infty & \text{if } \xi \in L^2(I) \setminus H^1(I) \end{cases}$$

over

$$K = \{\xi \in H^1(I) \mid \xi - Z \in [-\frac{\Delta}{2}, \frac{\Delta}{2}] \text{ on } I, \xi(c_i) = Z(c_i) + (-1)^{k(i)} \chi_i \frac{\Delta}{2}\},$$

where $i = l, r$ and $k(l) = 1, k(r) = 0$.

We note that it is easy to see that if the function Z is affine, the minimizer is affine as well. This is the case considered in this paper but, in general, even if $Z \in C^{1,1}(I)$, then it is known that the unique minimizer belongs to $Z \in C^{1,1}(I)$; see [23]. Also, we note that the derivative of minimizer $\bar{\xi}$ with C^1 smoothness is invariant with respect to shifts of Z by an additive constant. In view of these remarks, we write

$$\Lambda_{\chi_l \chi_r}^{Z'}(x; I, \Delta) = \bar{\xi}'(x) \quad \text{for } x \in I.$$

For $f \in C_P^2(U)$ and $\hat{x} \in U$, we define the *nonlocal curvature* $\Lambda_W(f)(\hat{x})$ as below. On the one hand, if $f'(\hat{x}) \notin P$, we set

$$\Lambda_W(f)(\hat{x}) = W''(f'(\hat{x}))f''(\hat{x})$$

as expected. On the other hand, if $p := f'(\hat{x}) \in P$, i.e. f is p -faceted at \hat{x} , then we set

$$\Lambda_W(f)(\hat{x}) = \Lambda_{\chi_l \chi_r}^\sigma(\hat{x}; I, \Delta)$$

where Δ is the jump in the derivative of W at p , i.e. $\Delta = \frac{d^+ W}{dq}(p) - \frac{d^- W}{dq}(p)$, $I = R(f, \hat{x})$, $\chi_l = \chi_l(f, \hat{x})$, $\chi_r = \chi_r(f, \hat{x})$, $\sigma(x) = 0$.

Let us denote, by $A_P(U_T)$, the set of all *admissible* functions on U_T , i.e. functions φ of the form $\varphi(t, x) = f(x) + g(t)$ where $f \in C_P^2(U)$ and $g \in C^1(0, T)$.

Definition 3.1 (Viscosity solutions). Let u be an $\bar{\mathbb{R}}$ -valued function on U_T . We say that u is a *viscosity subsolution* (resp. *supersolution*) of (3.1) if whenever $\varphi \in A_P(U_T)$ and $(\hat{t}, \hat{x}) \in U_T$ satisfy

$$\max_{U_T}(u^* - \varphi) = (u^* - \varphi)(\hat{t}, \hat{x}) = 0, \quad (\text{resp.} \quad \min_{U_T}(u_* - \varphi) = (u_* - \varphi)(\hat{t}, \hat{x}) = 0)$$

then the inequality

$$\varphi_t(\hat{t}, \hat{x}) + F(\Lambda_W(\varphi(\hat{t}, \cdot))(\hat{x})) \leq 0, \quad (\text{resp.} \quad \varphi_t(\hat{t}, \hat{x}) + F(\Lambda_W(\varphi(\hat{t}, \cdot))(\hat{x})) \geq 0)$$

holds. We say that u is a *viscosity solution* if u is both a viscosity subsolution and a viscosity supersolution.

We close this subsection with an explicit example of a viscosity solution starting from general initial data u_0 , not necessarily continuous.

Example 3.2. Let us consider the total variation flow (1.3) with the initial datum

$$u_0(x) = \begin{cases} 1 & \text{if } x \in (0, a), \\ 0 & \text{if } x \in (a, 1), \end{cases}$$

where $a \in (0, 1)$ is a given constant. One will realize that this problem (1.3), (1.2) has a solution of the form

$$u(t, x) = \begin{cases} 1 - \frac{2}{a}t & \text{if } t \leq a(1-a)/2, x \in (0, a), \\ \frac{2}{1-a}t & \text{if } t \leq a(1-a)/2, x \in (a, 1), \\ a & \text{if } t \geq a(1-a)/2. \end{cases}$$

In fact it is not difficult to check that u is a viscosity solution allowing a discontinuity.

3.2. Comparison and stability of viscosity solutions. We recall the comparison and stability results applicable in the present work.

Proposition 3.3. (Comparison, [23, Theorem 7]) *We assume (F1) and (F2) and we take $U = \mathbb{T}$. Let u and v respectively be a viscosity sub- and supersolution of (3.1) such that $u^* < +\infty$ and $v_* > -\infty$. If $u^*|_{t=0} \leq v_*|_{t=0}$ on \mathbb{T} , then $u^* \leq v_*$ in Q_T .*

Proposition 3.4. (Stability under extremum, [20, Theorems 4.1 and 5.4]) *Assume (F1) and (F2). Let S be a family of viscosity subsolutions (resp. supersolutions) of (3.1). Then,*

$$u(t, x) := \sup_{v \in S} v(t, x) \quad (\text{resp.} \quad u := \inf_{v \in S} v(t, x))$$

is a real-valued viscosity subsolution (resp. supersolution) of (3.1).

In summary, one is able to construct a unique solution of (3.1) with continuous initial data.

3.3. Related work. Pioneer work for the very singular diffusion equations including (3.1) is given by Giga-Giga in [16] and [17], which study the equations with spatially homogeneous external force. The authors introduced a notion of viscosity solutions, and established comparison, existence and stability of solutions in the sense of [16].

The definition of solutions in [16] looks different from one in the present paper. However, these two definitions are equivalent as far as one considers the equation

without a spatially inhomogeneous external force. Note that the value of Λ_W is determined explicitly when the external force is independent of the spatial variable. On the other hand, when the external force term depends on the spatial variable, one will encounter an obstacle problem. By considering the obstacle problem carefully, a comparison principle is given in [23], while a general existence result based on Perron method is given in [20].

As a further development of the viscosity solutions theory for singular diffusion equations, we point out that higher-dimensional total variation flows of a non-divergence form is studied in [22].

4. ENERGY SOLUTIONS ARE VISCOSITY SOLUTIONS

We prove here our main result.

Theorem 4.1. *If $u_0 \in BV(\mathbb{T})$, then the corresponding energy solution constructed in Theorem 2.5 is also a viscosity solution in the sense of Definition 3.1.*

The proof is divided in a number of steps.

4.1. Continuous initial data. We begin our analysis with the case of continuous initial data u_0 . Actually, we have seen in Proposition 2.7 that $u_0, u_{0,x} \in BV(\mathbb{T})$ guarantee that $u \in C^{\alpha, \alpha/2}(\mathbb{T} \times [0, T])$, for a positive α .

4.2. Preparation of the initial conditions. We prove here that we can approximate the BV data by continuous functions in an appropriate way.

Proposition 4.2. *Let us suppose that $u_0 \in BV(\mathbb{T})$. Then,*

(a) there exists an increasing sequence v_0^k , $k \in \mathbb{N}$ of continuous functions and such that $v_0^k \leq u_0$ a.e. and

$$\lim_{k \rightarrow 0} v_0^k(x) = u_0(x) \quad \text{for a.e. } x \in \mathbb{T};$$

(b) there exists a decreasing sequence w_0^k , $k \in \mathbb{N}$ of continuous functions and such that $w_0^k \geq u_0$ a.e. and

$$\lim_{k \rightarrow 0} w_0^k(x) = u_0(x) \quad \text{for a.e. } x \in \mathbb{T}.$$

The proof will be in a few steps. The first step is the Lemma below.

Lemma 4.3. *Let us suppose that $u : [a, b) \rightarrow \mathbb{R}$ is increasing, then there exists an increasing sequence of continuous functions $\{u_k\}_{k=1}^\infty$ such that $u_k(x) \leq u(x)$ and*

$$\lim_{k \rightarrow \infty} u_k(x) = u(x) \quad \text{for a.e. } x \in \mathbb{T}.$$

Proof. We extend u to \mathbb{R} by the following formula,

$$\tilde{u}(x) = \begin{cases} u(b^-), & \text{for } x \geq b, \\ u(x), & \text{for } x \in (a, b), \\ u(a), & \text{for } x \leq a. \end{cases}$$

Let us denote by ρ a standard mollifying kernel, whose support is $[-1, 1]$. We set

$$u_k(x) = (\tilde{u} * \rho_{1/k})(x - \frac{1}{k}).$$

Obviously, u_k are continuous. We have to check its properties. We notice, that $\text{supp } \rho \subset [-1, 1]$ implies

$$u_k(x) = \int_{\mathbb{R}} u(x - (y + 1)/k) \rho(y) dy \leq \int_{\mathbb{R}} u(x) \rho(y) dy = u(x).$$

By the same argument, we have

$$u_{k+1}(x) \geq u_k(x).$$

We know that $u * \rho_{1/k}$ converges to u in L^1 , hence there exists a subsequence, still denoted by u_k converging to u almost everywhere. Since the shift operator, $v(\cdot) \mapsto v(\cdot - h)$ is continuous in L^1 , we conclude that $u_k(x)$ converges to $u(x)$ for almost every $x \in [a, b)$. \square

Lemma 4.4. *Let us suppose that $f : [a, b] \rightarrow \mathbb{R}$ is such that f restricted to $[a, b]$ is increasing and $f(b) \in [f(a), f(b^-))$. Then, there exists an increasing sequence $\{f_k\}_{k=1}^\infty$ converging almost everywhere to f .*

Proof. We can take $u = f|_{[a, b)}$. Lemma 4.3 yields an increasing sequence u_k converging a.e. to u . We define it by the formula below,

$$f_k(x) = \begin{cases} u_k(x) & x \in [a, b - 1/k), \\ \ell_k(x - b) + f(b^-) & x \in [b - 1/k, b], \end{cases}$$

where

$$\ell_k = k(f(b^-) - u_k(b - 1/k)).$$

It is easy to see that f_k has the desired properties. \square

Lemma 4.5. *Let us suppose that $f : [a, b] \rightarrow \mathbb{R}$ is such that f restricted to $[a, b]$ is decreasing and $f(b) \in [f(b^-), f(a))$. Then, there exists a decreasing sequence $\{f_k\}_{k=1}^\infty$ converging almost everywhere to f .*

Proof. This result follows from Lemma 4.4 applied to $-f$. \square

Now, we are ready to prove Proposition 4.2. It is sufficient to establish (a) since (b) will follow from (a) applied to $-u_0$.

If we are given $u_0 \in BV(\mathbb{T})$, then we may choose its ‘good representative’ see [1, Theorem 3.28] and we define $f : [0, 1] \rightarrow \mathbb{R}$ by the following formula,

$$f(x) = \begin{cases} u_0(x) & x \in [0, 1) \\ u_0(0) & x = 1. \end{cases}$$

Of course, f has a finite variation over $(0, 1)$, hence $f = f^+ - f^-$, where f^\pm are increasing. By Lemmas 4.4 and 4.5, there exist $v_k^+, v_k^-, w_k^+, w_k^-$ such that

$$v_k^+ \leq f^+ \leq w_k^+, \quad v_k^- \leq f^- \leq w_k^-.$$

We set $v_k = v_k^+ - w_k^-$ and $w_k = w_k^+ - v_k^-$, these sequences monotonically converge to u_0 almost everywhere. Obviously, by the definition, we have

$$v_k \leq f^+ - f^- \leq w_k. \quad \square$$

4.3. BV initial data. Let us consider v^ϵ a unique continuous energy solution corresponding to v_0^ϵ , given by Proposition 4.2. Respectively, let w^ϵ be a unique continuous energy solution corresponding to w_0^ϵ , given by Proposition 4.2. We set

$$v = \sup v^\epsilon, \quad w = \sup w^\epsilon.$$

Lemma 4.6. *If v and w are defined above, then v and w are energy solutions.*

Proof. Due to monotonicity of families v^ϵ and w^ϵ , we see that v and w are in fact pointwise limits, so that we can use Proposition 2.8 implying that v and w are energy solutions. Hence, our claim follows. \square

Lemma 4.7. *v is a viscosity subsolution and w is a viscosity supersolution of (1.1), respectively.*

Proof. This lemma follows immediately from Proposition 3.4. \square

Proof of Theorem 4.1. We showed the following inequality for viscosity subsolution v and viscosity supersolution w ,

$$v \leq w.$$

At the same time v and w are energy solutions to (1.1) with the same initial condition u_0 . Uniqueness of energy solutions implies $v = w$, hence energy solutions to (1.1-1.2) are viscosity solutions to this equation. \square

Remark 4.8. We constructed in Theorem 2.5 solutions to (1.1)–(1.2) by the regularization method for data in BV. Then, we showed in Theorem 4.1 that these solutions are also viscosity solutions. In [32, Theorem 1] by the same method, we constructed solutions to (1.1)–(1.2) assuming more regular data, i.e. u_0 and $u_{0,x} \in BV$ and we showed that they are *almost classical*, [32, Theorem 3], see also [33]. Our Theorem 4.1 implies that all these types of solutions coincide, at least for more regular initial conditions. Finally, we point out that they are in fact solutions in the sense of the nonlinear semigroup theory, i.e. in a weak sense.

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